N64-17821
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CR-53358

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(NASA CR-53358;
TECHNICAL REPORT NO. 235-2)
JANUARY 15, 1964
17 p refs

OTS: $1.60 ph, $0.80 mf

12
OTS

prepared for
(NASA CONTRACT NO. NASw-682)
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON 25, D.C.

0147512

ALLIED RESEARCH ASSOCIATES, INC.,
VIRGINIA ROAD CONCORD, MASSACHUSETTS

SUMMARY

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Plastic buckling of thin shells under external pressure is considered using the shallow shell approach. Characteristic equations are obtained for simply supported type spherical plates and plastic buckling coefficients are determined for various values of a shell geometry parameter. AUTHOR

SYMBOLS

A_{ij}	=	plasticity coefficient matrix
$\bar{A}$	=	plasticity parameter = $(\frac{1}{2})\, A_{12}/A_{11}$
B	=	axial rigidity = $4E_s t/3$
D	=	flexural rigidity = $E_s t^3/9$
d	=	diameter of the base circle (dimensionless)
E_s	=	secant modulus
E_t	=	tangent modulus
F	=	stress function
k	=	buckling coefficient = $\sigma t\, d^2/\pi^2 D$
m	=	number of nodal circles
M	=	bending moment resultant per unit length
n	=	number of nodal lines
N	=	direct stress resultant per unit length
p	=	external pressure
r	=	radial coordinate (dimensionless)
R	=	radius of the sphere of which the cap is a part (dimensionless)
t	=	thickness of the shell (dimensionless)
u_ϕ, u_θ, w	=	dimensionless displacements in tangential, circumferential and normal directions
Z	=	dimensionless shell curvature parameter $= (d^2/Rt)\,(1 - \nu_p^2)^{\frac{1}{2}}$
α	=	dimensionless buckling stress = $(\sigma t/DA_{11})^{\frac{1}{2}}$
β	=	dimensionless shell geometry parameter $= (E_t t/DA_{11}^2 R^2)^{\frac{1}{2}}$

ϵ	=	direct strain variation
$\bar{\eta}$	=	plasticity reduction factor
ν_p	=	plastic Poisson ratio $= \frac{1}{2}$
σ	=	constant compressive stress at buckling $= pR/2t$
χ	=	curvature variation
∇^2	=	Laplacian operator $= \partial^2/\partial r^2 + (1/r)\,\partial/\partial r + (1/r^2)\partial^2/\partial\theta^2$

PLASTIC STABILITY OF SIMPLY SUPPORTED SPHERICAL PLATES UNDER EXTERNAL PRESSURE

Introduction

The use of spherical plates as pressure vessel closures in various types of vehicles has generated interest in the plastic stability of spherical plates under external pressure. The term spherical plate refers to those spherical elements which fall in the transition region between axially loaded circular flat plates where boundary conditions are significant and spherical shells where boundary conditions are insignificant.

The plastic stability of spherical shells has been investigated theoretically by Bijlaard[1], Gerard[2] and Lunchick[3]. Recently, Krenzke[4] conducted an experimental investigation to check these theories and found that the plasticity reduction factors were in good agreement when the discrepancy in elastic buckling coefficient was accounted for by an empirical correction.

In this paper, plastic stability theory of Ref. 2 is used to obtain plasticity reduction factors for spherical plates in the transition region between flat circular plates and spherical shells. Results are obtained for the axisymmetric and asymmetric buckling modes under external pressure based on an extension of the elastic stability of spherical plates presented in Ref. 5.

Governing Equations

In Ref. 2, Gerard has derived a Donnell type eigth order equation which is mainly applicable to the plastic buckling problem of cylindrical shells and the special case of that of a full sphere under external pressure. In the case of spherical plates, however, it is appropriate to use a shallow shell approach and obtain a coupled pair of Karman type equations.

The basic assumption of a shallow shell theory, namely that the vertical rise is small compared to the buckle wavelength, leads to simplifications in both the equilibrium and strain displacement relationships. In the following, the radius of the base circle representing the distance from the axis of the shell to the edge of the cap is taken as the unit of length and hence all lengths are essentially dimensionless.

The strain displacement relationships are:

$$\epsilon_\theta = u_\phi/r + (1/r)\partial u_\theta/\partial\theta + w/R \qquad \chi_\theta = (1/r)\partial w/\partial r + (1/r^2)\,\partial^2 w/\partial\theta^2 \tag{1}$$

$$\epsilon_\phi = \partial u_\phi/\partial r + w/R \qquad \chi_\phi = \partial^2 w/\partial r^2 \tag{2}$$

$$\epsilon_{\theta\phi} = 1/2[(1/r)\partial u_\phi/\partial\theta + r\,\partial/\partial r(u_\theta/r)] \quad \chi_{\theta\phi} = \partial/\partial r(1/r\,\partial w/\partial\theta) \tag{3}$$

In the above equations, ϵ, χ denote the direct strain and curvature variations, u_ϕ, u_θ, w the displacements in the meridional, circumferential and normal directions respectively.

The equilibrium equations for the buckling problem of a shallow spherical cap under external pressure, consistent with the strain displacement relations Eqs. (1) through (3), are:

$$(\partial/\partial r)(rN_\phi) + (\partial/\partial\theta)(N_{\phi\theta}) - N_\theta = 0 \tag{4}$$

$$(\partial/\partial r)(rN_{\phi\theta}) + \partial N_\theta/\partial\theta + N_{\phi\theta} = 0 \tag{5}$$

$$1/r\,[(\partial^2/\partial r^2)(rM_\phi) + 2(\partial^2/\partial r\partial\theta)(M_{\theta\phi}) - \partial M_\theta/\partial r + 2/r(\partial/\partial\theta)(M_{\theta\phi})]$$

$$+ (1/r^2)(\partial^2/\partial\theta^2)(M_\theta) + 1/R\,(N_\theta + N_\phi) + p + \sigma t\nabla^2 w = 0 \tag{6}$$

In Eqs. (4) to (6), N, M refer to the direct stress and moment resultants respectively; p is the external pressure, σ the constant compressive stress (= pR/2t) at buckling, R the radius of the sphere of which the cap is a part, t the thickness of the shell and ∇^2 the Laplacian operator.

The stress-strain relationship in the plastic range, utilizing deformation theory (Ref. 2), are:

$$N_\phi = BA_{11}(\epsilon_\phi + \bar{A}\epsilon_\theta) \qquad M_\phi = DA_{11}(\chi_\phi + \bar{A}\chi_\theta) \tag{7}$$

$$N_\theta = BA_{11}(\epsilon_\theta + \bar{A}\epsilon_\phi) \qquad M_\theta = DA_{11}(\chi_\theta + \bar{A}\chi_\phi) \tag{8}$$

$$N_{\phi\theta} = BA_{11}(1 - \bar{A})\epsilon_{\theta\phi} \qquad M_{\theta\phi} = DA_{11}(1 - \bar{A})\chi_{\theta\phi} \tag{9}$$

where $B = 4E_s t/3$ and $D = E_s t^3/9$; E_s being the secant modulus.

A_{11} is a component of the plasticity coefficient matrix A_{ij}, which for the spherical case has the following form (Ref. 2):

$$\begin{bmatrix} (3E_t/E_s + 1)/4 & (3E_t/E_s - 1)/2 & 0 \\ (3E_t/E_s - 1)/2 & (3E_t/E_s + 1)/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

where E_t is the tangent modulus and $\bar{A}$ is a plasticity parameter given by

$$\bar{A} = (1/2\, A_{12}/A_{11}) = (3E_t/E_s - 1)/(3E_t/E_s + 1) \tag{11}$$

The direct strain variations ϵ_θ, ϵ_ϕ, $\epsilon_{\theta\phi}$ are seen from Eqs. (1), (2) and (3) to satisfy a compatibility relationship

$$(1/r^2)(\partial^2 \epsilon_\phi/\partial\theta^2) - (1/r)(\partial\epsilon_\phi/\partial r) + (2/r)(\partial\epsilon_\theta/\partial r) + \partial^2 \epsilon_\theta/\partial r^2$$
$$- (1/r^2)(\partial^2/\partial r\partial\theta)(r2\epsilon_{\theta\phi}) = 1/R\, \nabla^2 w \tag{12}$$

It is readily seen that a stress function F can be introduced such that Eqs. (4) and (5) are identically satisfied, with the direct stress resultants being related as follows:

$$N_\phi = (1/r)(\partial F/\partial r) + (1/r^2)(\partial^2 F/\partial\theta^2);\; N_\theta = \partial^2 F/\partial r^2;\; N_{\theta\phi} = -(\partial/\partial r)(1/r\,\partial F/\partial\theta) \tag{13}$$

By use of Eq. (13), the governing equations of the problem reduce to the solving of Eqs. (6) and (12), with F being introduced into them.

By making use of Eqs. (10) to (12); (1) to (3) and (13), Eqs. (12) and (6) can be written in the following form:

$$\nabla^4 F = (E_t t/A_{11} R)\nabla^2 w \tag{14}$$

$$DA_{11}\nabla^4 w + 1/R\,\nabla^2 F + p + \sigma t\nabla^2 w = 0 \tag{15}$$

These then are the governing equations of the spherical plate problem.

Before we deal with Eqs. (14) and (15) as such, it is advantageous to study the flat circular plate and full sphere cases first. The spherical plate solution will readily yield the results of the circular plate case by letting the appropriate term containing the height of the shell vanish. It is also possible to obtain the solution for the full sphere as a singular case of the same equations.

Flat Circular Plate

For a flat circular plate under axial compression with $R \to \infty$ in Eqs. (14) and (15), we obtain the following governing equation:

$$\nabla^4 w + (\sigma t/DA_{11})\nabla^2 w + (p/DA_{11}) = 0 \tag{16}$$

If we let p=0 in Eq. (16), we have

$$\nabla^2(\nabla^2 + a^2)w = 0 \tag{17}$$

where $a^2 = \sigma t/DA_{11}$). The general solution of Eq. (17), with the requirement that w, $1/r\ \partial w/\partial r$ and $\partial^2/\partial r^2$ be finite at r=0, can be written as

$$w = \sum_{n=0}^{\infty} [C_{on} r^n + C_{1n} J_n(ar)] \cos n\theta \tag{18}$$

In Eq. (18) n represents the number of nodal lines on the deformed surface.

For a simply supported edge, we obtain the characteristic equation for the buckling coefficient a as:

$$aJ_n(a) - (1 - \bar{A})\,J_{n+1}(a) = 0;\ \text{with } n \geq 0 \tag{19}$$

The lowest root of Eq. (19) for n = 0, the axisymmetric case, provides the critical buckling stress for a given E_t/E_s ratio. In order to separate out the plasticity effects it is useful to define a plasticity reduction factor $\bar{\eta}$, following Ref. 2, defined as

$$\bar{\eta} = (\sigma/D)_{plastic} \, / \, (\sigma/D)_{elastic} \tag{20}$$

In Table 1, numerical results for Eq. (19) with n = 0 are given in terms of $\bar{\eta}$ for different E_t/E_s ratios.

Table 1: Plasticity Reduction Factors for Simply Supported Flat Circular Plates

E_t/E_s	$\bar{\eta}$
1.0	1.0
0.75	0.765
0.50	0.526
0.25	0.278
0	0

Full Sphere

Eqs. (14) and (15) are transformed, after operating with ∇^2, into the following:

$$\nabla^6 w + (\sigma t/DA_{11}) \, \nabla^4 w + (E_t t/A_{11}^2 DR^2) \nabla^2 w = 0 \tag{21}$$

$$\nabla^6 F + (\sigma t/DA_{11}) \nabla^4 F + (E_t t/A_{11}^2 \, DR^2) \nabla^2 F = -(E_t t/DA_{11}^2 R) p \tag{22}$$

We assume a constant stress state throughout the region under pressure given by $\nabla^2 F = -pR$ so that Eq. (22) is satisfied identically. Then a suitable form for displacement w would be

$$w = C_n J_n(kr) \cos n\theta \tag{23}$$

By substituting Eq. (23) into Eq. (21) and obtaining a minimum condition for $(\sigma t/DA_{11})$ we find that

$$(\sigma t/DA_{11})_{min} = 2(E_t t/DA_{11}^2 R^2)^{\frac{1}{2}} \tag{24}$$

or

$$\sigma_{cr} = [3\ (1 - \nu_e^2)]^{-\frac{1}{2}}\ \eta(Et/R) \tag{25}$$

where

$$\eta = [(1 - \nu_e^2) / (1 - \nu^2)]^{\frac{1}{2}} (E_s/E)(E_t/E_s)^{\frac{1}{2}} \tag{26}$$

In Eq. (26) ν is the current Poisson ratio as in Ref. 2. It is readily seen that by taking $\eta = 1$ in Eq. (25) we obtain the elastic critical stress.

It is clear that the expression for σ_{cr} in Eq. (25) is independent of n in the expression for w in Eq. (23). Hence for both axi- and asymmetric modes, we obtain the same critical stress for the sphere.

Spherical Plate

In the case of a spherical plate, where both curvature and boundary affect the buckling stress, a simple assumption on the stress state cannot be made as in the case of the sphere. Hence we must consider the general solutions of Eq. (21) and (22) which can be written conveniently as

$$\nabla^2(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)w = 0 \tag{27}$$

$$\nabla^2(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)F = - k_1^2 k_2^2 Rp \tag{28}$$

where

$$k_1^2, k_2^2 = \alpha^2/2 \pm [(\alpha^2/2)^2 - \beta^2]^{\frac{1}{2}} \tag{29}$$

$$\alpha^2 = \sigma t/DA_{11}, \; \beta^2 = E_t t/DA_{11}^2 R^2 \tag{29}$$

It is seen from Eq. (24) that the case of full sphere corresponds to $\alpha^2/2 = \beta$ or $k_1 = k^2$ in Eq. (29). Hence for the spherical plate we need consider only $k_1 \neq k_2$.

The solutions of Eqs. (27) and (28) can now be written as:

$$w = \sum_{n=0}^{\infty} [C_{on} r^n + C_{1n} J_n (k_1 r) + C_{2n} J_n (k_2 r)] \cos n\theta \tag{30}$$

$$F = -pRr^2/4 - (E_t t/A_{11} R) \sum_{n=0}^{\infty} [A_{on} r^n + (C_{1n}/k_1^2) J_n (k_1 r)$$

$$+ (C_{2n}/k_2^2) J_n (k_2 r)] \cos n\theta \tag{31}$$

where the finiteness of w, $1/r\ \partial w/\partial r$, $\partial^2 w/\partial r^2$, F, $1/r\ \partial F/\partial r$ and $\partial^2 F/\partial r^2$ at $r = 0$ is taken into account.

For a simply-supported type edge fixture we can write the following boundary conditions, as discussed in Ref. (5), at the edge $r = 1$;

$$w = 0 \qquad M_\phi = 0 \qquad \nabla^2 F = -pR \tag{32}$$

With the above boundary conditions, the characteristic equation for the buckling stress is written as follows:

$$(k_2^2 - k_1^2) J_n (k_1) J_n (k_2) + (1 - \bar{A}) [k_1 J_n (k_2) J_{n+1} (k_1) - k_2 J_n (k_1) J_{n+1} (k_2)] = 0 \tag{33}$$

for $n \geq 0$.

Eq. (33) can be solved for k_1 (or k_2) for a given value of β. Each non-trivial root for a given n corresponds to an increased number of nodal circles starting from the single nodal circle corresponding to the axisymmetric case of n = 0. Hence, we can characterize the results of Eq. (33) by their m, n numbers.

Numerical Results

Figure 1 shows a plot of the results of Eq. (33) for various values obtained by the use of a digital computer for E_t/E_s ratio of 1, 0.75, 0.50 and 0.25.

For convenience of comparison with the results from cylindrical shells the coordinates have been re-defined. A buckling coefficient $k = \sigma t d^2/\pi^2 D$, where d is the base diameter, and a shell parameter Z, given by $Z = (d^2/Rt)(1 - \nu_p^2)^{\frac{1}{2}}$ have been used where α and β are related to k and Z in the following manner:

$$k = \sigma t d^2/\pi^2 D = (d^2/\pi^2) A_{11} \alpha^2 = 4\alpha^2 A_{11}/\pi^2 \tag{34}$$

$$Z = d^2 (1 - \nu_p^2)^{\frac{1}{2}}/Rt = 4(1 - \nu_p^2)^{\frac{1}{2}}/Rt = A_{11}(E_t/E_s)^{-\frac{1}{2}}(2\sqrt{3})\beta \tag{35}$$

ν_p being the fully plastic value of Poisson ratio $= \frac{1}{2}$

In Eqs. (34) and (35), since all lengths are normalized with respect to the radius of the base circle, d = 2.

Thus for the spherical case of $\alpha^2/2 = \beta$, we have

$$k = (4\sqrt{3}/\pi^2)(E_t/E_s)^{\frac{1}{2}} Z \tag{36}$$

The elastic case ($E_t/E_s = 1$) corresponds to k = .702 Z. For $E_t/E_s = 0.25$ the minimum line is given by k =.351 Z.

The k - Z plots of Figure 1, show the curves that correspond to various buckling modes indicated by m, n specifications, for E_t/E_s ratios of 1 and 0.25. It is a distinctive feature of the spherical plate problem that there is a single minimum line $k = (4\sqrt{3}/\pi^2)(E_t/E_s)^{\frac{1}{2}}Z$, for the value of $E_t/E_s \leq 1$, corresponding to both axi- and asymmetric modes as contrasted with the cylindrical problem, Ref. (6), where for the plastic case there are two minimum lines depending upon whether the governing mode is axi- or asymmetric.

To compare the plastic and elastic behavior using the plasticity reduction faction defined in Eq. (20), values of $\bar{\eta}$ have been determined for the different modes by using the data presented in Figure 1. These data are presented in Figure 2 and Figure 3. For large Z values, where the straight line becomes the best approximation for the buckling stress as seen from Figure 1, $\bar{\eta}$ is seen from Eq. (36) to be

$$\bar{\eta} = (E_t/E_s)^{\frac{1}{2}}$$

For other boundary conditions, the determined $\bar{\eta}$ can be used with the correct elastic buckling stress coefficient k.

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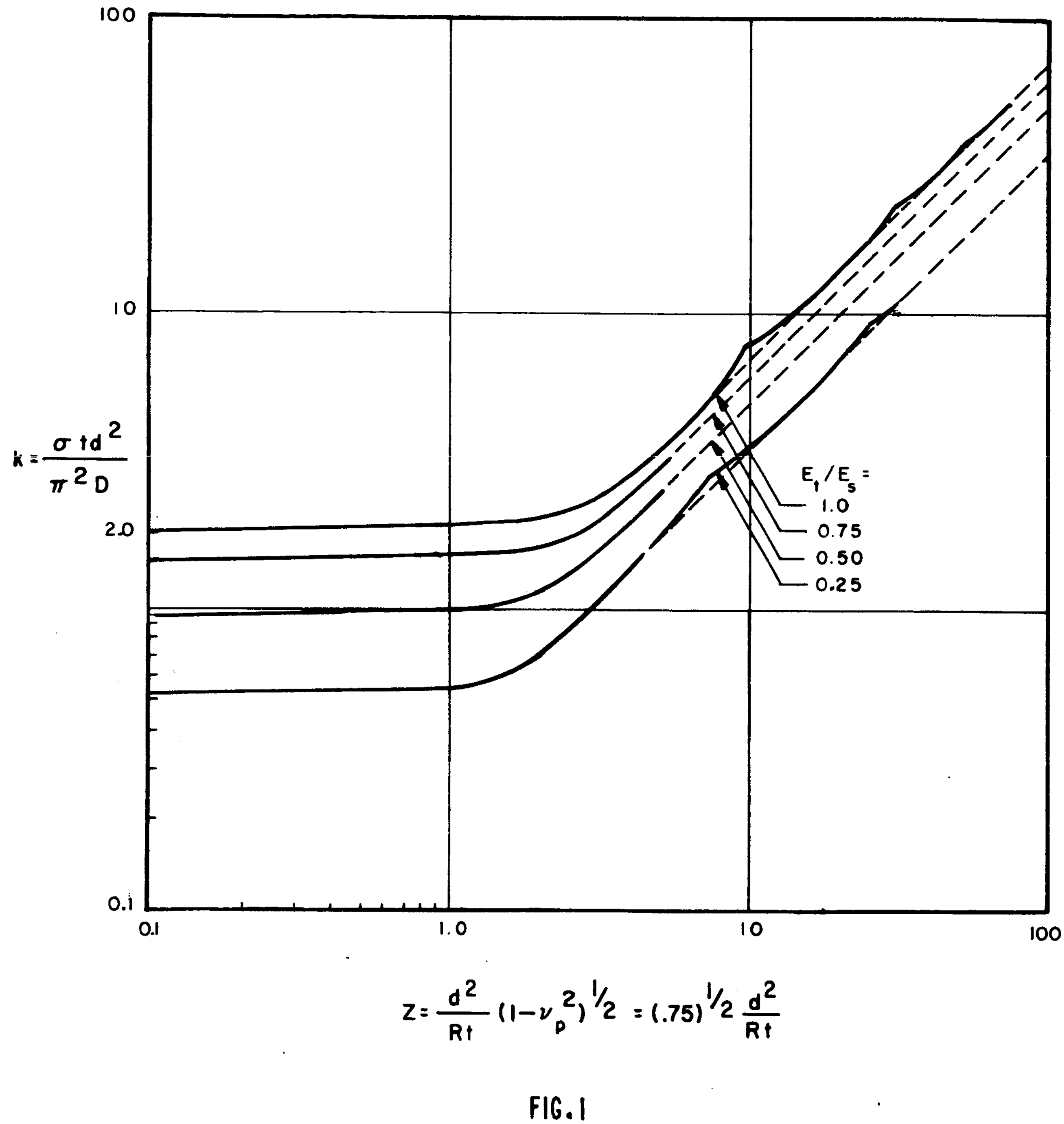

FIG. 1

PLASTIC BUCKLING COEFFICIENT AS A FUNCTION OF Z

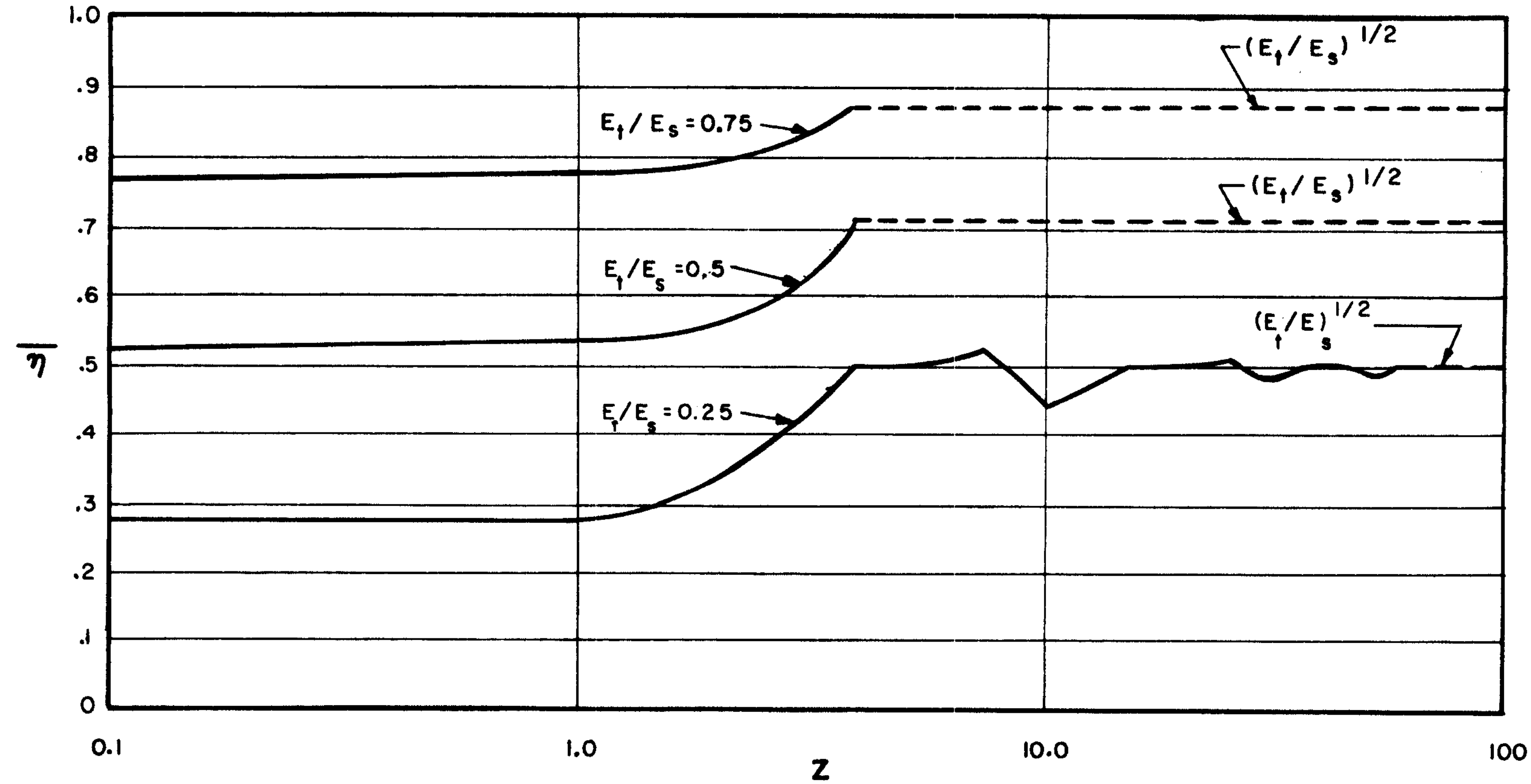

FIG. 2

PLASTICITY REDUCTION FACTOR AS A FUNCTION OF Z

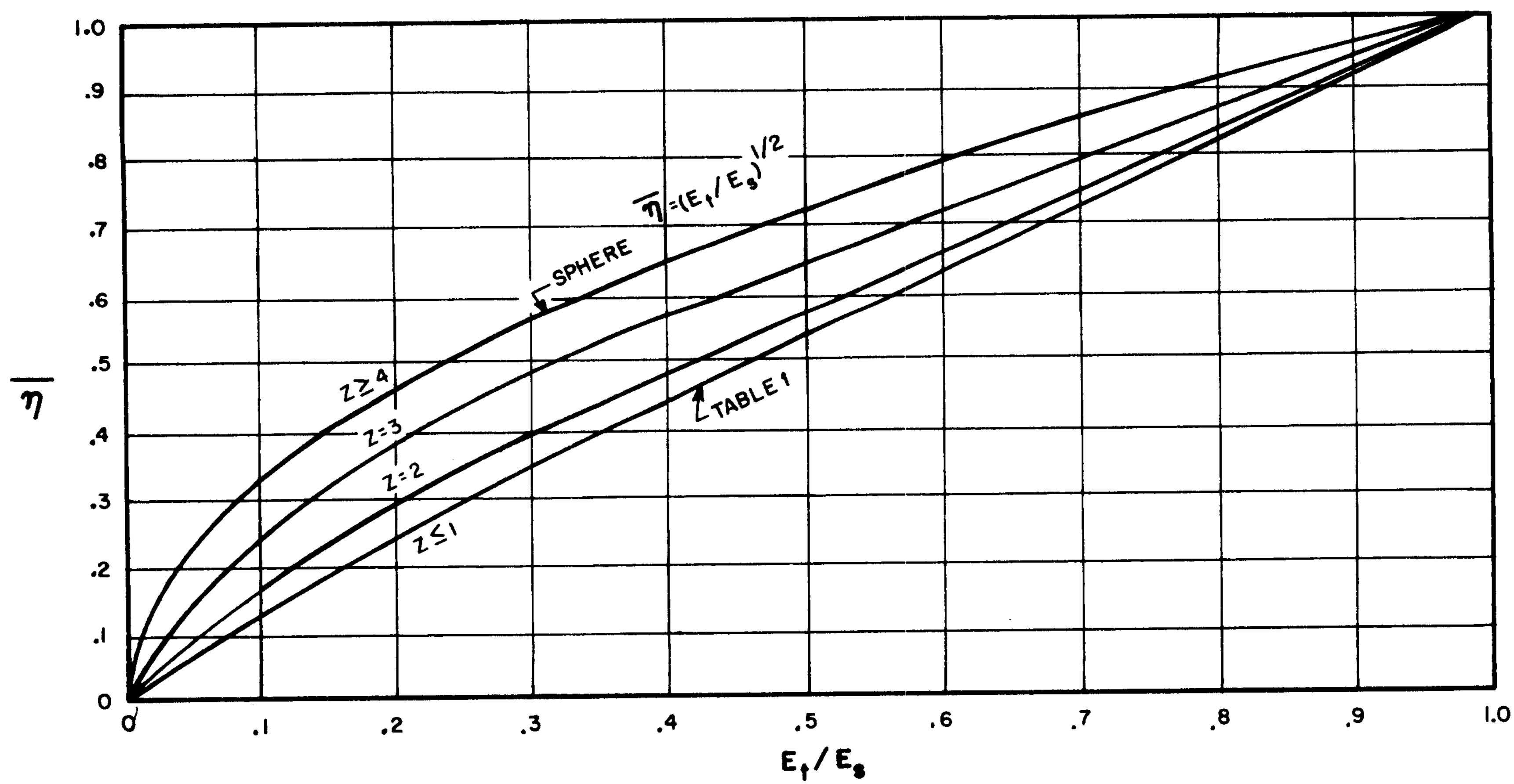

FIG. 3

PLASTICITY REDUCTION FACTOR AS A FUNCTION OF E_t/E_s